B.Sc (Physics) Part-I,

Paper-I, Group-B

The Hamiltonian function and conservation of energy:

Lagrangian L in terms of generalized coordinates $q_1, q_2, --- q_n$ and generalized velocities $\dot{q}_1, \dot{q}_2, -- \dot{q}_n$ i's

$$L = L(q_1, q_2, --- q_n; \dot{q}_1, \dot{q}_2, ---, \dot{q}_n; t) \quad \text{--- (1)}$$

we write above equation, for simplicity, as

$$L = L(q_k; \dot{q}_k; t) \quad \text{--- (2)} \quad \text{where} \quad k = 1, 2, --- n$$

Now taking total derivative of L

$$\frac{dL}{dt} = \sum_k \frac{\partial L}{\partial q_k}\frac{dq_k}{dt} + \sum_k \frac{\partial L}{\partial \dot{q}_k}\frac{d\dot{q}_k}{dt} + \frac{\partial L}{\partial t} \quad \text{--- (3)}$$

Lagrange's equations for conservative system is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \quad \text{--- (4)}$$

using (4) in (3)

$$\frac{dL}{dt} = \sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\dot{q}_k + \sum_k \frac{\partial L}{\partial \dot{q}_k}\frac{d\dot{q}_k}{dt} + \frac{\partial L}{\partial t}$$

or $$\frac{dL}{dt} = \sum_k \frac{d}{dt}\left(\dot{q}_k \frac{\partial L}{\partial \dot{q}_k}\right) + \frac{\partial L}{\partial t}$$

or $\frac{d}{dt}\left[\sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L\right] = -\frac{\partial L}{\partial t}$ —— (5)

Since Lagrangian of a closed system cannot be an explicity function of time, Therefore, we write

$$\frac{\partial L}{\partial t} = 0$$

and we write eq. (5) as

$$\frac{d}{dt}\left[\sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L\right] = 0$$

The quantity in the square brackets must be constant in time. We denote this constant by H which is called the Hamiltonian.

$$H = \sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L = \text{constant}$$

or $\boxed{H = \sum_k p_k \dot{q}_k - L}$ = const. $\left\{ \frac{\partial L}{\partial \dot{q}_k} = p_k \rightarrow \text{generalized momentum} \right.$

Thus if L is not explicit function of time, then $\frac{\partial L}{\partial t} = 0$ gives

$$\frac{dH}{dt} = 0$$

or $H = \sum_k p_k \dot{q}_k - L = \text{Constant}$

In special case H is equal to total energy E of the system. For conservative system, $\frac{\partial V}{\partial \dot{q}_k} = 0$ and we have $p_k = \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k}(T - V) = \frac{\partial T}{\partial \dot{q}_k}$

$\therefore H = \sum_k p_k \dot{q}_k - L = \sum_k \frac{\partial T}{\partial \dot{q}_k}\dot{q}_k - L = 2T - L = 2T - (T - V)$ $\left\{ \because \sum_k \frac{\partial T}{\partial \dot{q}_k}\dot{q}_k = 2T \right.$ (Home work)

or $\boxed{H = T + V = E}$